On some extremal problems in spaces of harmonic functions

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Abstract

We give solutions to some extremal problems involving distance function in mixed normed spaces of harmonic functions on the unit ball in $\mathbb{R}^n$.

Key words:
Bergman spaces, distance estimates, extremal problems, harmonic functions. **2000 AMS subject classifications**: 30H20.

† Supported by Ministry of Science, Serbia, project OI174017

1 Introduction

The classical duality approach to distance problems in spaces of analytic functions is based on a functional analysis result:

$$\mathrm{dist}_X(f, Y) = \inf_{g \in Y} \|f - g\|_X = \sup_{\phi \in Y^\perp, \|\phi\| \le 1} |\phi(f)|,$$

where $Y^\perp \subset X^\star$ is the annihilator of a subspace Y of a normed space X. This method was employed in the case of Hardy spaces, [8], [2]. More recently, extremal problems in the Bergman spaces were considered in [7]. The aim of this paper is to obtain extremal distance results in spaces of harmonic functions on the unit ball $\mathbb{B} \subset \mathbb{R}^n$ using direct methods introduced by R. Zhao, [15], [14]. We note that spaces of harmonic functions on the unit ball $\mathbb{B}$ were extensively studied, see [1], [3] and [6] and references therein.

In this paper letter C designates a positive constant which can change its value even in the same chain of inequalities. Given real expressions A and B, we write $A \lesssim B$ if there is a constant $C \ge 0$ such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$ then we write $A \asymp B$. The integer part of a real number α is denoted by $[\alpha]$, and the fractional part of α is denoted by $\{\alpha\} = \alpha - [\alpha]$.

We use standard notation: $\mathbb{S} = \partial\mathbb{B}$ is the unit sphere in $\mathbb{R}^n$, for $x \in \mathbb{R}^n$ we have $x = rx'$, where $r = |x| = \sqrt{\sum_{j=1}^n x_j^2}$ and $x' \in \mathbb{S}$. Normalized

Lebesgue measure on $\mathbb{B}$ is denoted by $dx = dx_1 \dots dx_n = r^{n-1} dr dx'$ so that $\int_{\mathbb{B}} dx = 1$. The space of all harmonic functions on $\mathbb{B}$ is denoted by $h(\mathbb{B})$.

We consider harmonic weighted Bergman spaces $A^p_\alpha(\mathbb{B})$ on $\mathbb{B}$ defined for $0 \le \alpha < \infty$ and $0 < p \le \infty$ by

$$A^p_\alpha(\mathbb{B}) = \left\{ f \in h(\mathbb{B}) : \|f\|_{p,\alpha} = \left(\int_{\mathbb{B}} |f(rx')|^p (1-r)^\alpha r^{n-1} dr dx' \right)^{1/p} < \infty \right\},$$

$$A^\infty_\alpha(\mathbb{B}) = \left\{ f \in h(\mathbb{B}) : \|f\|_{\infty,\alpha} = \sup_{x \in \mathbb{B}} |f(x)|(1-|x|)^\alpha < \infty \right\}.$$

The spaces $A^p_\alpha = A^p_\alpha(\mathbb{B})$ are Banach spaces for $1 \le p \le \infty$ and complete metric spaces for $0 < p < 1$.

We are going to use some results on spherical harmonics from [13] and A^p_α spaces from [1]. Let $Y^{(k)}_j$ be the spherical harmonics of order k, $1 \le j \le \alpha_k$, on $\mathbb{S}$. Next,

$$Z^{(k)}_{x'}(y') = \sum_{j=1}^{\alpha_k} Y^{(k)}_j(x') \overline{Y^{(k)}_j(y')},$$

are zonal harmonics of order k. Note that the spherical harmonics $Y^{(k)}_j$, $(k \ge 0, 1 \le j \le \alpha_k)$ form an orthonormal basis of $L^2(\mathbb{S}, dx')$. Every $f \in h(\mathbb{B})$ has an expansion

$$f(x) = f(rx') = \sum_{k=0}^{\infty} r^k Y^{(k)}(x'),$$

where $Y^{(k)} = \sum_{j=1}^{\alpha_k} c^j_k Y^k_j$. Using this expansion one defines, as in [1], fractional derivative of order $t \in \mathbb{R}$ of $f \in h(\mathbb{B})$ by the following formula:

$$\mathcal{D}^t f(rx') = \sum_{k=0}^{\infty} r^k \frac{\Gamma(k+t+n/2)}{\Gamma(k+n/2)\Gamma(t+n/2)} Y^{(k)}(x').$$

Let

$$Q_\alpha(x,y) = 2 \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1+k+n/2)}{\Gamma(\alpha+1)\Gamma(k+n/2)} r^k \rho^k Z^{(k)}_{x'}(y'), \qquad x = rx', y = \rho y' \in \mathbb{B}$$

for $\alpha > -1$, where Γ is the Euler's Gamma function. $Q_\alpha(x,y)$ is harmonic in each of the variables separately, $Q_\alpha(x,y) = \overline{Q_\alpha(y,x)}$ and $\|Q_\alpha(x,\cdot)\|_{A^p_\alpha} = \|Q_\alpha(\cdot,y)\|_{A^p_\alpha} = 1$, see [1]. In fact, it is a harmonic Bergman kernel for the space A^p_α, we have the following theorem from [1], see also [4]:

Theorem 1.1 *Let $p \ge 1$ and $\alpha \ge 0$. Then for every $f \in A^p_\alpha$ and $x \in \mathbb{B}$ we have*

$$f(x) = \int_0^1 \int_{\mathbb{S}} (1-\rho^2)^\alpha Q_\alpha(x,y) f(\rho y') \rho^{n-1} d\rho dy', \qquad y = \rho y'.$$

The following lemma from [1] gives estimates for this kernel.

Lemma 1.1 *1. Let $\alpha > 0$. Then, for $x = rx', y = \rho y' \in \mathbb{B}$ we have*

$$|Q_\alpha(x,y)| \le C\frac{(1-r\rho)^{-\{\alpha\}}}{|\rho x - y'|^{n+[\alpha]}} + \frac{C}{(1-r\rho)^{1+\alpha}}.$$

2. Let $\beta > -1$.

$$\int_{\mathbb{S}} |Q_\beta(rx',y)|dx' \le \frac{C}{(1-r\rho)^{1+\beta}}, \qquad |y| = \rho.$$

3. Let $m > n-1$, , $0 \le r < 1$ and $y' \in \mathbb{S}$. Then

$$\int_{\mathbb{S}} \frac{dx'}{|rx'-y'|^m} \le \frac{C}{(1-r)^{m-n+1}}.$$

Next we note that there is a variant of Theorem 1.1 for $0 < p < 1$, Theorem 1.2 below, which is weaker in the sense that it does not provide integral representation, but stronger in the sense that allows more general weights. First we introduce the class of admissible weight functions.

Let S be the class of all functions $\omega(t) \ge 0$, $0 < t < 1$ such that

$$m_\omega \le \frac{\omega(\lambda r)}{\omega(r)} \le M_\omega, \qquad \lambda \in [q_\omega, 1], 0 < r < 1.$$

for some constants $m_\omega, M_\omega, q_\omega \in (0,1)$. An example of a function ω in S is

$$\omega(t) = t^\alpha \left(\log\left(\frac{C}{t}\right)\right)^\beta, \qquad \alpha > -1, \beta > 0.$$

For $\omega \in S$ we set $\alpha_\omega = \log m_\omega / \log q_\omega$ and $\beta_\omega = -\log M_\omega / \log q_\omega$. To each $\omega \in S$ and $0 < p \le 1$ we associate a weighted harmonic Bergman space

$$A^p_\omega(\mathbb{B}) = \left\{ f \in h(\mathbb{B}) : \int_0^1 \int_{\mathbb{S}} |f(rx')|^p \omega(1-r) r^{n-1} dx' dr < \infty \right\}.$$

A characterization of functions in $A^p_\omega(\mathbb{B})$ was obtained in [11], in order to formulate this result it is convenient to introduce some notation. We recall polar coordinates of $x \in \mathbb{R}^n$: $x_1 = r\cos\phi_1$, $x_2 = r\sin\phi_1\cos\phi_2$,...,$x_{n-1} = r\sin\phi_1 \ldots \sin\phi_{n-2}\cos\phi_{n-1}$, $x_n = r\sin\phi_1\ldots\sin\phi_{n-1}$ where $r > 0$, $0 \le \phi_i < \pi$ for $1 \le i \le n-2$ and $-\pi < \phi_{n-1} \le \pi$. Given $k \ge 0$, $0 \le l_i < 2^k$, $1 \le i < n-2$ and $-2^k \le l_{n-1} < 2^k$ we define $\Delta_{k;l_1,\ldots,l_{n-1}}$ as the set of all $x \in \mathbb{B}$ such that the polar coordinates of x satisfy: $1 - 2^{-k} \le r < 1 - 2^{-k-1}$, $\pi l_i/2^k \le \phi_i < \pi(l_i+1)/2^k$ for $1 \le i \le n-2$ and $\pi l_{n-1}/2^k < \phi_{n-1} \le \pi(l_{n-1}+1)/2^k$. We denote the Poisson kernel for the unit ball by $P(x,y)$, $x \in \mathbb{B}$, $y \in \mathbb{S}$. Since it is harmonic in x we can introduce, for $\alpha > 1$,

$$P_\alpha(x,y) = \mathcal{D}_x^{\alpha-1} P(x,y).$$

Theorem 1.2 ([11]) *Let $\omega \in S$, $0 < p \le 1$ and $\alpha > \frac{\alpha_\omega + n}{p} - n$. Then a function $f \in h(\mathbb{B})$ belongs to A^p_ω if and only if*

$$\int_{\mathbb{B}} (1 - |y|^2)^\alpha P_\alpha(x, y) f(y) d\mu(y) < \infty$$

for every Borel measure μ on $\mathbb{B}$ such that

$$\sum_{k=0}^{\infty} \sum_{l_1=0}^{2^k-1} \cdots \sum_{l_{n-1}=-2^k}^{2^k-1} [\mu(\Delta_{k;l_1,\dots,l_{n-1}})]^p \omega(|\Delta_{k;l_1,\dots,l_{n-1}}|^{1/n}) |\Delta_{k;l_1,\dots,l_{n-1}}|^{1-p} < \infty.$$

We record here for later use the following well known result.

Lemma 1.2 *Let $\beta > -1$, $s > -1$, $\gamma > 0$ and $0 < p \le 1$. Then for every increasing function $G(r)$, $0 \le r < 1$ we have the following estimate:*

$$\int_0^1 \left(\int_0^1 \frac{G(r)(1-r)^\beta dr}{(1 - r\rho)^\gamma} \right)^p (1-\rho)^s d\rho \;\le\; C \int_0^1 \int_0^1 \frac{G(r)^p (1-r)^{\beta p + p - 1}}{(1 - r\rho)^{\gamma p}} (1-\rho)^s d\rho dr,$$

where C depends only on β, s, γ and p.

For $f \in h(\mathbb{B})$ we define p-integral means of f by

$$M_p(f, r) = \left(\int_{\mathbb{S}} |f(rx')|^p dx' \right)^{1/p}, \qquad 0 \le r < 1$$

for $0 < p < \infty$, with the usual modification for $p = \infty$. The function $M_p(f, r)$ is increasing in $0 \le r < 1$ for $p \ge 1$ because $|f|^p$ is subharmonic for $1 \le p < \infty$. We are going to consider the following spaces of harmonic functions, for $0 < p < \infty$, $1 \le q < \infty$ and $\alpha > 0$:

$$B^{\infty,q}_\alpha = \left\{ f \in h(\mathbb{B}) : \|f\|_{B^{\infty,q}_\alpha} = \sup_{0 \le r < 1} M_q(f, r)(1-r)^\alpha < \infty \right\},$$

$$B^{p,q}_\alpha = \left\{ f \in h(\mathbb{B}) : \|f\|_{B^{p,q}_\alpha} = \left(\int_0^1 M_q^p(f, r)(1-r)^{\alpha p - 1} dr \right)^{1/p} < \infty \right\},$$

$$B^{p,\infty}_\alpha = \left\{ f \in h(\mathbb{B}) : \|f\|_{B^{p,\infty}_\alpha} = \left(\int_0^1 M_\infty^p(f, r)(1-r)^{\alpha p - 1} dr \right)^{1/p} < \infty \right\}.$$

These spaces have obvious (quasi)-norms, with respect to these (quasi)-norms they are Banach spaces or complete metric spaces. The following elementary lemma is used below.

Lemma 1.3 *If $\beta > 0$, the for every increasing function $\phi(r)$, $0 \le r < 1$ we have*

$$\sup_{0 \le r < 1} \phi(r)(1-r)^\beta \le \beta \int_0^1 \phi(r)(1-r)^{\beta-1} dr.$$

Proof. For every $r_0 \in [0,1)$ we have

$$\int_0^1 \phi(r)(1-r)^{\beta-1} dr \ge \int_{r_0}^1 \phi(r)(1-r)^{\beta-1} dr$$
$$\ge \phi(r_0) \int_{r_0}^1 (1-r)^{\beta-1} dr = \frac{1}{\beta}\phi(r_0)(1-r_0)^\beta. \quad \square$$

Using this lemma and a fact that $M_q(f,r)$ is increasing for $q \ge 1$ we obtain the following estimate, valid for $q \ge 1$, $0 < p < \infty$ and $\alpha > 0$:

$$\|f\|^p_{B^{\infty,q}_\alpha} = \sup_{r<1} M^p_q(f,r)(1-r)^{\alpha p} \le \alpha p \int_0^1 M^p_q(f,r)(1-r)^{\alpha p-1} dr = \alpha p \|f\|^p_{B^{p,q}_\alpha}.$$

This shows that $B^{p,q}_\alpha$ is continuously embedded in $B^{\infty,q}_\alpha$ for the above range of parameters. Also, it is elementary that $B^{\infty,1}_\alpha \subset A^1_\alpha$. These two embedding results allow us to use the integral representation from Theorem 1.1 to functions f in $B^{\infty,1}_\alpha \supset B^{p,1}_\alpha$, $p > 0$. Also, the last inclusion leads to a natural problem of finding $\mathrm{dist}_{B^{\infty,1}_\alpha}(f, B^{p,1}_\alpha)$, which we investigate in the next section.

2 Main results

This section is devoted to formulations and proofs of our main results on extremal problems in spaces of harmonic functions. In order to state our first theorem we introduce the following notation: for $\epsilon > 0$, $\alpha > 0$ and $f \in h(\mathbb{B})$ we set

$$L_{\epsilon,\alpha}(f) = \left\{ r \in [0,1) : (1-r)^\alpha \int_{\mathbb{S}} |f(rx')| dx' \ge \epsilon \right\}.$$

Theorem 2.1 *Let $1 \le p < \infty$, $\alpha > 0$. Set, for $f \in B^{\infty,1}_\alpha$,*

$$s_1 = s_1(f) = \mathrm{dist}_{B^{\infty,1}_\alpha}(f, B^{p,1}_\alpha),$$
$$s_2 = s_2(f) = \inf\left\{ \epsilon > 0 : \int_0^1 \chi_{L_{\epsilon,\alpha}(f)}(r)(1-r)^{-1} dr < \infty \right\}.$$

Then $s_1 \asymp s_2$, with constants involved depending only on p, α and n.

Proof. First we prove that $s_1 \geq s_2$. Assume to the contrary that $s_1 < s_2$. Then there are $\epsilon > \epsilon_1 > 0$ and $f_1 \in B^{p,1}_\alpha$ such that $\|f - f_1\|_{B^{\infty,1}_\alpha} \leq \epsilon_1$ and $\int_0^1 \chi_{L_{\epsilon,\alpha}}(r)(1-r)^{-1}dr = \infty$. Hence

$$\begin{aligned}(1-r)^\alpha \int_{\mathbb{S}} |f_1(rx')|dx' &\geq (1-r)^\alpha \int_{\mathbb{S}} |f(rx')|dx' \\ &\quad - \sup_{0\leq r<1} (1-r)^\alpha \int_{\mathbb{S}} |f(rx') - f_1(rx')|dx' \\ &\geq (1-r)^\alpha \int_{\mathbb{S}} |f(rx')|dx' - \epsilon_1,\end{aligned}$$

and therefore we have, for $r \in L_{\epsilon,\alpha}(f)$, the following estimate:

$$(1-r)^\alpha \int_{\mathbb{S}} |f_1(rx'|dx' \geq (1-r)^\alpha \int_{\mathbb{S}} |f(rx')|dx' - \epsilon_1 \geq \epsilon - \epsilon_1. \tag{1}$$

However, this leads to a contradiction:

$$\begin{aligned}(\epsilon - \epsilon_1)^p \int_0^1 \chi_{L_{\epsilon,\alpha}(f)}(r)(1-r)^{-1}dr &\leq \int_0^1 M_1^p(f_1, r)(1-r)^{\alpha p-1}dr \\ &= \|f_1\|^p_{B^{p,1}_\alpha} < \infty.\end{aligned}$$

Next we prove $s_1 \leq Cs_2$. We fix $f \in B^{\infty,1}_\alpha$ and $\epsilon > 0$ such that the integral appearing in the definition of $s_2(f)$ is finite. Then, using integral representation from Theorem 1.1:

$$\begin{aligned}f(x) &= \int_0^1 \int_{\mathbb{S}} (1-\rho^2)^\alpha Q_\alpha(x,y) f(\rho y')\rho^{n-1} d\rho dy' \\ &= \int_{L_{\epsilon,\alpha}(f)} \int_{\mathbb{S}} (1-\rho^2)^\alpha Q_\alpha(x,y) f(\rho y')\rho^{n-1} d\rho dy' \\ &+ \int_{I\setminus L_{\epsilon,\alpha}(f)} \int_{\mathbb{S}} (1-\rho^2)^\alpha Q_\alpha(x,y) f(\rho y')\rho^{n-1} d\rho dy' \\ &= f_1(x) + f_2(x),\end{aligned} \tag{2}$$

where $I = [0,1)$. To complete the proof it suffices to prove

$$\|f_1\|_{B^{p,1}_\alpha} \leq C_\epsilon \|f\|_{B^{\infty,1}_\alpha} \tag{3}$$

and

$$\|f_2\|_{B^{\infty,1}_\alpha} \leq C\epsilon. \tag{4}$$

In proving (3) it suffices to consider the case $p = 1$, since the general case then follows from the estimate $\|f_1\|_{B^{p,1}_\alpha} \leq C\|f_1\|_{B^{1,1}_\alpha}$ from [9]. Now we have, using Fubini's theorem and Lemma 1.1:

$$\|f_1\|_{B^{1,1}_\alpha} = \int_{\mathbb{B}} (1-\rho)^{\alpha-1} |f_1(\rho x')| d\rho dx'$$

$$
\begin{aligned}
&\le \int_{\mathbb{B}}(1-\rho)^{\alpha-1}\int_{L_{\epsilon,\alpha}(f)}\int_{\mathbb{S}}\left(\frac{C}{|1-r\rho|^{1+\alpha}}+\frac{C(1-r\rho)^{-\{\alpha\}}}{|r\rho x'-y'|^{n+[\alpha]}}\right)\\
&\times\ (1-r^2)^{\alpha}|f(ry')|drdy'd\rho dx'\\
&\le C\int_{L_{\epsilon,\alpha}(f)}\int_{x'\in\mathbb{S}}(1-r^2)^{\alpha}|f(ry')|\int_0^1\frac{(1-\rho)^{\alpha-1}}{(1-r\rho)^{1+\alpha}}d\rho dy'dr\\
&\le C\int_{L_{\epsilon,\alpha}(f)}(1-r)^{\alpha-1}\int_{y'\in\mathbb{S}}|f(ry')|dy'dr\\
&\le C\|f_1\|_{B_{\alpha}^{\infty,1}}\int_{L_{\epsilon,\alpha}(f)}(1-r)^{-1}dr\\
&= C_{\epsilon}\|f_1\|_{B_{\alpha}^{\infty,1}}.
\end{aligned}
$$

Now we turn to (4). For every $0\le\rho<1$ we have, using Lemma 1.1, the following estimate:

$$
\begin{aligned}
(1-\rho)^{\alpha}\int_{\mathbb{S}}|f_2(\rho x')|dx' &\le C(1-\rho)^{\alpha}\int_{\mathbb{S}}\int_{I\setminus L_{\epsilon,\alpha}(f)}\int_{\mathbb{S}}|f(ry')|(1-r^2)^{\alpha}\\
&\times\left(\frac{(1-\rho r)^{-\{\alpha\}}}{|r\rho x'-y'|^{n+[\alpha]}}+\frac{1}{(1-r\rho)^{1+\alpha}}\right)dy'drdx'
\end{aligned}
$$

Now, using Lemma 1.1 again and Fubini's theorem we obtain, taking into account definition of the set $L=L_{\epsilon,\alpha}(f)$

$$
\begin{aligned}
(1-\rho)^{\alpha}\int_{\mathbb{S}}|f_2(\rho x')|dx' &\le C(1-\rho)^{\alpha}\int_{I\setminus L}\int_{\mathbb{S}}|f(ry')|(1-r^2)^{\alpha}\frac{dy'dr}{(1-r\rho)^{1+\alpha}}\\
&\le C\epsilon(1-\rho)^{\alpha}\int_{I\setminus L}(1-r)^{-\alpha}(1-r^2)^{\alpha}\frac{dr}{(1-r\rho)^{1+\alpha}}\\
&\le C\epsilon,
\end{aligned}
$$

which ends the proof of Theorem 2.1. □

Our next results is an analogue of the previous theorem for the case $0<p\le 1$.

Theorem 2.2 *Let $0<p\le 1$, $\alpha>0$ and $t>\alpha-1$. For $f\in B_{\alpha}^{\infty,1}$ we set*

$$\hat{s}_1(f)=\mathrm{dist}_{B_{\alpha}^{\infty,1}}(f,B_{\alpha}^{p,1}),$$

$$\hat{s}_2(f)=\inf\left\{\epsilon>0:\int_0^1\left(\int_0^1\chi_{L_{\epsilon,\alpha}(f)}(r)\frac{(1-r)^{t-\alpha}}{(1-r\rho)^{t+1}}dr\right)^p(1-\rho)^{p\alpha-1}d\rho<\infty\right\}.$$

Then $\hat{s}_1(f)\asymp\hat{s}_2(f)$, with constants involved depending only on α, n, p and t.

We note that for $p=1$ we have, using Fubini's theorem, $\hat{s}_2(f)=s_2(f)$. Therefore, for $p=1$, the above two theorems give the same answer to the distance problem.

Proof. Our argument here is similar to the proof of the previous theorem. We first prove that $\hat{s}_2(f) \le \hat{s}_1(f)$. Assume that $\hat{s}_1(f) < \hat{s}_2(f)$. Then there are $0 < \epsilon_1 < \epsilon$ and $f_1 \in B^{p,1}_\alpha$ such that

$$\int_0^1 \left(\int_0^1 \chi_{L_{\epsilon,\alpha}(f)}(r) \frac{(1-r)^{t-\alpha}}{(1-r\rho)^{t+1}} dr \right)^p (1-\rho)^{p\alpha-1} d\rho = \infty \tag{5}$$

and $\|f - f_1\|_{B^{\infty,1}_\alpha} \le \epsilon_1$. We can repeat the argument given at the beginning of the proof of Theorem 2.1 to arrive at estimate (1) for $r \in L_{\epsilon,\alpha}(f)$, and this implies

$$\chi_{L_{\epsilon,\alpha}(f)}(r) \le (\epsilon - \epsilon_1)^{-q} \left(\int_{\mathbb{S}} |f_1(rx')| dx' \right)^q (1-r)^{\alpha q}, \qquad 0 < q < \infty. \tag{6}$$

Now Lemma 1.2 can be applied to $G(r) = M(f_1, r)$, $\beta = t$, $s = \alpha p - 1$ and $\gamma = t + 1$ to obtain, using (6) with $q = 1$,

$$\begin{aligned}
&\int_0^1 \left(\int_0^1 \frac{(1-r)^{t-\alpha} \chi_{L_{\epsilon,\alpha}(f)}(r) dr}{(1-r\rho)^{t+1}} \right)^p (1-\rho)^{\alpha p-1} d\rho \\
\le& C \int_0^1 \int_0^1 \left(\int_{\mathbb{S}} |f_1(rx')dx' \right)^p \frac{(1-r)^{(t+1)p-1}}{(1-r\rho)^{(t+1)p}} (1-\rho)^{\alpha p-1} dr d\rho \\
\le& C \|f_1\|_{B^{p,1}_\alpha}.
\end{aligned}$$

But this gives a contradiction with (5).

Next we prove $\hat{s}_1(f) \le C\hat{s}_2(f)$. Let us fix $f \in B^{\infty,1}_\alpha$ and $\epsilon > 0$ such that the integral appearing in the definition of $\hat{s}_2(f)$ is finite. We use the same decomposition $f = f_1 + f_2$ that appear in (2), and the same reasoning as in the proof of Theorem 2.1 gives $\|f_2\|_{B^{\infty,1}_\alpha} \le C\epsilon$. Therefore it remains to prove that $f_1 \in B^{p,1}_\alpha$. The following chain of inequalities

$$\begin{aligned}
\|f_1\|^p_{B^{p,1}_\alpha} =& \int_0^1 (1-\rho)^{\alpha p-1} \Big(\int_{\mathbb{S}} |f_1(\rho x')| dx' \Big)^p d\rho \\
\le& C \int_0^1 (1-\rho)^{\alpha p-1} \Big(\int_{L_{\epsilon,\alpha}(f)} \int_{\mathbb{S}} (1-r^2)^t \int_{\mathbb{S}} \Big(\frac{1}{|r\rho x' - y'|^{n+t}} + \\
&+ \frac{(1-r\rho)^{-\{t\}}}{|r\rho x' - y'|^{n+[t]}} \Big) |f(ry')| dr dy' \Big)^p d\rho \\
\le& \left(\sup_{0 \le r < 1} (1-r)^\alpha \int_{\mathbb{S}} |f(rx')| dx' \right)^p \\
&\times \int_0^1 \left(\int_0^1 \chi_{L_{\epsilon,\alpha}(f)}(r) \frac{(1-r)^{t-\alpha} dr}{(1-r\rho)^{t+1}} \right)^p (1-\rho)^{\alpha p-1} d\rho \\
\le& C_\epsilon \|f\|^p_{B^{\infty,1}_\alpha}.
\end{aligned}$$

completes the proof. □

For $f \in h(\mathbb{B})$, $\epsilon > 0$ and $\alpha > 0$ we define

$$\hat{L}_{\epsilon,\alpha}(f) = \{r : 0 \leq r < 1, M_\infty(f,r)(1-r)^\alpha \geq \epsilon\}.$$

Using Lemma 1.3 one easily checks that $B^{p,\infty}_\alpha \subset A^\infty_\alpha$. The following theorem is similar to the previous one, the difference is that space $B^{\infty,1}_\alpha$ is replaced by A^∞_α.

Theorem 2.3 *Let $\alpha > 0$ and $1 \leq p < \infty$. For $f \in A^\infty_\alpha$ we set*

$$\tilde{s}_1(f) = \mathrm{dist}_{A^\infty_\alpha}(f, B^{p,\infty}_\alpha)$$

and

$$\tilde{s}_2(f) = \inf\left\{\epsilon > 0 : \int_0^1 \chi_{\hat{L}_{\epsilon,\alpha}(f)}(r)(1-r)^{-1}dr < \infty\right\}.$$

Then $\tilde{s}_1(f) \asymp \tilde{s}_2(f)$, with constants involved depending only on α, p and n.

Proof. We use the same ideas as above. We first prove $\tilde{s}_2(f) \leq \tilde{s}_1(f)$. Assume $\tilde{s}_1(f) < \tilde{s}_2(f)$. Then there are $0 < \epsilon_1 < \epsilon$ and $f_1 \in B^{p,\infty}_\alpha$ such that $\|f - f_1\|_{A^\infty_\alpha} \leq \epsilon_1$ and

$$\int_0^1 \chi_{\hat{L}_{\epsilon,\alpha}(f)}(r)(1-r)^{-1}dr = \infty. \tag{7}$$

Since $\|f - f_1\|_{A^\infty_\alpha} \leq \epsilon_1$ we have

$$\begin{aligned}(1-r)^\alpha|f_1(x)| &= |(1-r)^\alpha f(x) - (1-r)^\alpha[f(x) - f_1(x)]| \\ &\geq (1-r)^\alpha|f(x)| - \epsilon_1\end{aligned}$$

for every $x \in \mathbb{B}$, $|x| = r$. Hence, for $r \in \hat{L}_{\epsilon,\alpha}(f)$, we have

$$M_\infty(f_1,r)(1-r)^\alpha \geq M_\infty(f,r)(1-r)^\alpha - \epsilon_1 \geq \epsilon - \epsilon_1,$$

which implies

$$\chi_{\hat{L}_{\epsilon,\alpha}(f)}(r) \leq \frac{1}{(\epsilon-\epsilon_1)^p} M^p_\infty(f_1,r)(1-r)^{\alpha p}.$$

Since $f_1 \in B^{p,\infty}_\alpha$ this easily leads to a contradiction with (7).

Next we prove $\tilde{s}_1(f) \leq C\tilde{s}_2(f)$. Let us fix $f \in A^\infty_\alpha$ and $\epsilon > 0$ such that the integral appearing in the definition of $\tilde{s}_2(f)$ is finite. We use the same decomposition $f = f_1 + f_2$ that appears in (2), and it suffices to prove the following two estimates:

$$\|f_2\|_{A^\infty_\alpha} \leq C\epsilon, \tag{8}$$

$$\|f_1\|_{B^{p,\infty}_\alpha} \leq C_{\epsilon,p}\|f\|_{A^\infty_\alpha}. \tag{9}$$

Set $\hat{L} = \hat{L}_{\epsilon,\alpha}(f)$. Since $|f(\rho y')| \leq \epsilon(1-\rho)^{-\alpha}$ for $\rho \in I \setminus \hat{L}$ we have, using Lemma 1.1, for every $x = rx' \in \mathbb{B}$

$$\begin{aligned} |f_2(x)| &\leq C\epsilon \int_{I\setminus\hat{L}} \int_{\mathbb{S}} (1-\rho^2)^\alpha \left(\frac{(1-r\rho)^{-\{\alpha\}}}{|\rho x - y'|^{n+[\alpha]}} + \frac{1}{(1-r\rho)^{1+\alpha}} \right) \rho^{n-1} \frac{d\rho dy'}{(1-\rho)^\alpha} \\ &\leq C\epsilon \int_0^1 \frac{d\rho}{(1-r\rho)^{1+\alpha}} + C\epsilon \int_0^1 (1-r\rho)^{-\{\alpha\}} \int_{\mathbb{S}} \frac{dy'}{|\rho x - y'|^{n+[\alpha]}} d\rho \\ &\leq C\epsilon(1-r)^{-\alpha}, \end{aligned}$$

which gives estimate (8). Since $|f(y)| \leq \|f\|_{A^\infty_\alpha}(1-\rho)^{-\alpha}$, $|y| = \rho$, we have for any $x = rx' \in \mathbb{B}$:

$$\begin{aligned} |f_1(x)| &\leq \|f\|_{A^\infty_\alpha} \int_{\hat{L}} \int_{\mathbb{S}} (1-\rho^2)^\alpha |Q_\alpha(x, \rho y')| (1-\rho)^{-\alpha} \rho^{n-1} d\rho dy' \\ &\leq C\|f\|_{A^\infty_\alpha} \int_{\hat{L}} \int_{\mathbb{S}} |Q_\alpha(x, \rho y')| d\rho dy' \\ &\leq C\|f\|_{A^\infty_\alpha} \int_{\hat{L}} \frac{d\rho}{(1-r\rho)^{1+\alpha}}, \end{aligned}$$

where we used Lemma 1.1 again. We proved that $M_\infty(f_1, r) \leq C\|f\|_{A^\infty_\alpha}\phi(r)$, where

$$\phi(r) = \int_{\hat{L}} \frac{d\rho}{(1-r\rho)^{1+\alpha}}, \qquad 0 \leq r < 1. \tag{10}$$

Desired estimate (9) will be established once we prove that

$$\psi(r) = (1-r)^\alpha \phi(r) \in L^p(I, (1-r)^{-1} dr).$$

An application of Fubini's theorem gives

$$\begin{aligned} \int_0^1 (1-r)^\alpha \int_{\hat{L}} \frac{d\rho}{(1-r\rho)^{1+\alpha}} \frac{dr}{1-r} &= \int_{\hat{L}} \int_0^1 \frac{(1-r)^{\alpha-1} dr}{(1-r\rho)^{1+\alpha}} d\rho \\ &\leq C_\alpha \int_{\hat{L}} \frac{d\rho}{1-\rho} < \infty \end{aligned}$$

by the condition imposed on ϵ. This proves that $\psi \in L^1(I, (1-r)^{-1}dr)$. Since $(1-r)^\alpha(1-\rho) < (1-r\rho)^{1+\alpha}$ for $r, \rho \in I$ we have

$$\psi(r) = \int_{\hat{L}} \frac{(1-r)^\alpha(1-\rho)}{(1-r\rho)^{1+\alpha}} \frac{d\rho}{1-\rho} \leq \int_{\hat{L}} \frac{d\rho}{1-\rho} = C,$$

and this proves that $\psi \in L^\infty(I, (1-r)^{-1}dr)$. But then it clearly follows that $\psi \in L^p(I, (1-r)^{-1}dr)$ for all $1 \leq p < \infty$. □

The above theorem can be extended to cover the case $0 < p < 1$, this is the content of the next theorem.

Theorem 2.4 *Let* $\alpha > 0$ *and* $0 < p \leq 1$. *For* $f \in A_\alpha^\infty$ *we set*

$$\overline{s}_1(f) = \mathrm{dist}_{A_\alpha^\infty}(f, B_\alpha^{p,\infty}),$$

$$\overline{s}_2(f) = \inf\left\{\epsilon > 0 : \int_0^1 \left(\int_{\hat{L}_{\epsilon,\alpha}(f)} \frac{dr}{(1-r\rho)^{\alpha+1}}\right)^p (1-\rho)^{\alpha p-1} d\rho < \infty\right\}.$$

Then $\overline{s}_1(f) \asymp \overline{s}_2(f)$, *with constants involved depending only on* α, p *and* n.

Proof. As usual, we prove $\overline{s}_2(f) \leq \overline{s}_1(f)$ arguing by contradiction: if $\overline{s}_1(f) < \overline{s}_2(f)$ then there are $0 < \epsilon_1 < \epsilon$ and $f_1 \in B_\alpha^{p,\infty}$ such that $\|f - f_1\|_{A_\alpha^\infty} \leq \epsilon_1$ and

$$J_\epsilon = \int_0^1 \left(\int_{\hat{L}_\epsilon} \frac{dr}{(1-r\rho)^{\alpha+1}}\right)^p (1-\rho)^{\alpha p-1} d\rho = \infty, \tag{11}$$

where $\hat{L}_\epsilon = \hat{L}_{\epsilon,\alpha}(f)$. Since

$$\begin{aligned} (1-r)^\alpha M_\infty(f_1, r) &\geq (1-r)^\alpha M_\infty(f, r) - (1-r)^\alpha M_\infty(f - f_1, r) \\ &\geq (1-r)^\alpha M_\infty(f, r) - \epsilon_1, \end{aligned}$$

we have $(1-r)^\alpha M_\infty(f_1, r) \geq \epsilon - \epsilon_1$ for $r \in \hat{L}_\epsilon$, or

$$\chi_{\hat{L}_\epsilon}(r) \leq (\epsilon - \epsilon_1)^{-1}(1-r)^\alpha M_\infty(f_1, r).$$

Therefore, using Lemma 1.2, we obtain:

$$\begin{aligned} J_\epsilon &\leq C \int_0^1 \left(\int_0^1 \frac{(1-r)^\alpha M_\infty(f_1, r)}{(1-r\rho)^{\alpha+1}} dr\right)^p (1-\rho)^{\alpha p-1} d\rho \\ &\leq C \int_0^1 \int_0^1 \frac{(1-r)^{p(\alpha+1)-1} M_\infty(f_1, r)^p}{(1-r\rho)^{(\alpha+1)p}} (1-\rho)^{\alpha p-1} dr d\rho \\ &\leq C \int_0^1 M_\infty(f_1, r)^p (1-r)^{\alpha p-1} dr = C\|f_1\|_{B_\alpha^{p,\infty}} < \infty \end{aligned}$$

which contradicts (11). Next we turn to the estimate $\overline{s}_1(f) \leq C\overline{s}_2(f)$, using the same technique as in the previous theorems: we fix $f \in A_\alpha^\infty$ and choose $\epsilon > 0$ such that the integral appearing in the definition of $\overline{s}_2(f)$ is finite. Again we use decomposition $f = f_1 + f_2$ from (2) and the argument presented in the proof of Theorem 2.3 gives $\|f_2\|_{A_\alpha^\infty} \leq C\epsilon$, see derivation of (8). Now it clearly suffices to prove $f_1 \in B_\alpha^{p,\infty}$, in fact we show that

$$\|f_1\|_{B_\alpha^{p,\infty}} \leq C_\epsilon \|f\|_{A_\alpha^\infty}. \tag{12}$$

As in the proof of the previous theorem we have $M_\infty(f_1, r) \leq C\|f\|_{A_\alpha^\infty}\phi(r)$, where ϕ is defined by (10). Therefore (12) follows from the following

$$\int_0^1 \phi(r)^p (1-r)^{\alpha p-1} dr = \int_0^1 \left(\int_{\hat{L}_\epsilon} \frac{d\rho}{(1-r\rho)^{1+\alpha}}\right)^p (1-r)^{\alpha p-1} dr = C_\epsilon < \infty. \quad \square$$

Let us introduce, for $0 < r < 1$, $\alpha > -1$ and $f \in h(\mathbb{B})$

$$A_\alpha(f,r) = \int_{|w|\le r|} |f(w)|(1-|w|)^\alpha dw.$$

Next we consider, for $\alpha > -1$, $\beta > 0$ and $0 < p < \infty$ the spaces

$$M^\alpha_\beta(\mathbb{B}) = \left\{ f \in h(\mathbb{B}) : \|f\|_{\alpha,\beta} = \sup_{0\le r<1} (1-r)^\beta A_\alpha(f,r) < \infty \right\},$$

$$M^\alpha_{p,\beta}(\mathbb{B}) = \left\{ f \in h(\mathbb{B}) : \|f\|^p_{\alpha,\beta;p} = \int_0^1 (1-r)^{\beta p-1} A_\alpha(f,r)^p dr < \infty \right\}.$$

The spaces M^α_β and $M^\alpha_{p,\beta}$, $1 \le p < \infty$, are Banach spaces and the spaces $M^\alpha_{p,\beta}$ are complete metric spaces for $0 < p < 1$. Using Lemma 1.3 it is readily seen that $B^\alpha_{p,\beta} \subset B^\alpha_\beta$. We note that the above spaces were investigated in the case of dimension two by several authors, for example in [5] and [10]. In order to investigate distance problem in these spaces we introduce, for $\epsilon > 0$, $\alpha > -1$, $\beta > 0$ and $f \in h(\mathbb{B})$ the set

$$\tilde{L}^\alpha_{\epsilon,\beta}(f) = \left\{ r \in (0,1) : (1-r)^\beta A_\alpha(f,r) \ge \epsilon \right\}.$$

Theorem 2.5 *Let $p \ge 1$, $\alpha > -1$, $\beta > 0$. We set, for $f \in M^\alpha_\beta$,*

$$t_1(f) = \mathrm{dist}_{M^\alpha_\beta}(f, M^\alpha_{p,\beta}),$$

$$t_2(f) = \inf \left\{ \epsilon > 0 : \int_0^1 \chi_{\tilde{L}^\alpha_{\epsilon,\beta}(f)}(r)(1-r)^{-1} dr < \infty \right\}.$$

Then $t_1(f) \asymp t_2(f)$, with constants involved depending only on α, β, p and n.

We omit the proof of this theorem since it is quite similar to the proofs of all theorems presented in this section.

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